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Numerical investigation of the two– dimensional time–dependent diffusion equation using Radial basis functions

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Abstract

In this paper, we investigate the numerical solution of the two–dimensional time–dependent diffusion equation with non-local and mixed Neumann–Dirichlet boundary conditions. In the discretization process, the backward Euler as well as Crank–Nicolson schemes and radial basis function (RBF) collocation method are respectively used to discretize time derivative and spatial derivative terms. The accuracy and applicability of the presented methods are illustrated and compared by solving two examples.

1. Introduction

RBFs interpolation was first introduced by Roland Hardy in 1968 [2]. Later Franke [12] obtained that the RBFs interpolations were accurate, comparing the many methods he tested. Kansa [4, 5] first used RBFs for estimation of partial differential equations (PDEs). However, in recent years the RBF method has been considered as a novel numerical approach to solve the various ordinary, partial and integral differential equations [1, 8, 9, 16]. This method is meshless and it can be extended to multi–dimensional problems. The main idea of the meshless methods is that they can gain an accurate and stable solution of integral equations or partial differntial equations with differential boundary conditions with a set of scattered data without using any mesh [3].

Parabolic initial boundary value problems in one dimension which involve non–local conditions have been studied by several authors. Also, the two–dimensional diffusion equation with nonlocal boundary conditions arises in many important applications in heat transfer [7], control theory [17], medical science [18] and so on. The numerical study of the two–dimensional time–dependent diffusion equation with non–local conditions and Dirichlet boundary conditions has been presented in [6]. Dehghan [10] has proposed a second–order finite difference scheme for the numerical solution of a boundary value problem with Neumann's boundary conditions. Also, Abbasbandy and Shirzadi [14] applied the MLPG method to solve this type of equation as well. A meshless method for twoâ€“dimensional diffusion equation with an integral condition has been investigated in [15].

In this paper, we consider two–dimensional time–dependent diffusion equation [16]

$$\frac{\partial u}{\partial t} = \Delta u, \qquad \Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{1}$$

with initial condition

$$u(x, y, 0) = f(x, y), \qquad 0 \le x, y \le 1, \tag{2}$$

and mixed Neumann–Dirichlet conditions

$$\frac{\partial u(x,y,t)}{\partial x}|_{x=0} = g_0(y, t), \qquad 0 \le y \le 1, 0 < t \le 1, \tag{3}$$

$$\frac{\partial u(x,y,t)}{\partial x}|_{x=1} = g_1(y, t), \qquad 0 \le y \le 1, 0 < t \le 1, \tag{4}$$

$$u(x, 1, t) = h_1(x, t), \qquad 0 \le x \le 1, 0 < t \le 1, \tag{5}$$

$$u(x, 0, t) = h_0(x, t)\mu(t), \qquad 0 \le x \le 1, 0 < t \le 1, \tag{6}$$

and non–local condition

$$\int_0^1 \int_0^1 u(x, y, t)dxdy = m(t), \qquad 0 < t \le 1, \tag{7}$$

where f, g_0, g_1, h_0, h_1 and m are known functions, while the functions u and μ must be determined.

The rest of the paper is organized as follows: In the second section, the RBFs approximation for solving PDEs by using the collocation method has been presented. In the third section, we obtain a time-discrete scheme using Euler finite difference and Crank–Nicolson techniques. In the fourth section, the non–local boundary value problem has been implemented by RBFs. We give two computational results of numerical experiments with the RBF method, to support theoretical discussion in the fifth section. Finally, a conclusion is given in the last section.

2. RBFs approximation

2.1 Definition of radial basis functions

RBFs depend on the Euclidean distance $r = \| \mathbf{x} - \mathbf{x}^* \|$ between the fixed point $\mathbf{x}^*$, and a generic point $\mathbf{x}$. These functions may be generically represented in the form $\phi(r)$. In Table 1 some globally supported RBFs are listed which are commonly employed in the literature. The positive constant c in RBFs named shape parameter.

Table 1: Some well-known RBFs, $c > 0$ [16, 21-28].

Name of function	Abbreviation	Formula
Multiquadric	MQ	$\varphi(r) = \sqrt{c^2 + r^2}$
Inverse multiquadric	IMQ	$\varphi(r) = \frac{1}{\sqrt{c^2 + r^2}}$
Inverse quadric	IQ	$\varphi(r) = \frac{1}{c^2 + r^2}$
Gaussian	GA	$\varphi(r) = \exp(-cr^2)$

2.2 RBF approximation based on Kansa's method

The d-dimesional function $u(\mathbf{x})$, $u: \mathbb{R}^d \to \mathbb{R}$, to be interpolated and approximated by an RBF as

$$u(\mathbf{x}) \simeq u_N(\mathbf{x}) = \sum_{i=1}^{N} \lambda_i \phi_i(r) = \Phi^T(r)\Lambda, \tag{8}$$

where

$$\phi_i(r) = \phi(\| \mathbf{x} - \mathbf{x}_i \|),$$
$$\Phi^T(r) = [\phi_1(r), \phi_2(r), \dots, \phi_N(r)],$$
$$\Lambda = [\lambda_1, \lambda_2, \dots, \lambda_N]^T, \qquad (9)$$

$\mathbf{x}$ is the input and $\{\lambda_i\}_{i=1}^N$ are the set of coefficients to be determined. Choosing N collocation nodes $\{\mathbf{x}_i\}_{i=1}^N$ gives the approximation function of $u(\mathbf{x})$ after solving the following system

$$u(\mathbf{x}) \simeq u_N(\mathbf{x}_j) = \sum_{i=1}^N \lambda_i \phi_i(r_j), \qquad j = 1,2, \dots, N.$$

We now discuss Kansa's collocation method. Assume we are given a domain $\Omega \subset \mathbb{R}^d$, and a partial differential equation operator of the form

$$L[u](\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \qquad (10)$$

with boundary conditions

$$B[u](\mathbf{x}) = g(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega. \qquad (11)$$

Then we approximate u by radial basis functions as

$$u(\mathbf{x}) = \sum_{i=1}^N \lambda_i \phi(\| \mathbf{x} - \mathbf{x}_i \|). \qquad (12)$$

The simplest possible setting is shown in Eq. (12). The collocation matrix is constructed by matching the differential equation (10) and the boundary conditions (11) at the collocation nodes $\{\mathbf{x}_j\}_{j=1}^N$ of the form

$$A = \begin{bmatrix} L[\Phi] \\ B[\Phi] \end{bmatrix}, \qquad (13)$$

where the blocks generated as follow

$$L[\Phi]_{ji} = L[\phi](\| \mathbf{x}_j - \mathbf{x}_i \|), \qquad \mathbf{x}_j \in \Omega^\circ, \mathbf{x}_i \in \Omega, \qquad (14)$$
$$B[\Phi]_{ji} = B[\phi](\| \mathbf{x}_j - \mathbf{x}_i \|), \qquad \mathbf{x}_j \in \partial\Omega, \mathbf{x}_i \in \Omega, \qquad (15)$$

where Ω° is interior of Ω. Here we identify the collocation points same as center points. Consequently, the system $A\Lambda = \mathbf{c}$ gives the RBF solution of PDE, while $\mathbf{c}$ is defined as the form

$$\mathbf{c} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}, \qquad (16)$$

where $\mathbf{f} = [f(\mathbf{x}_j)], \mathbf{x}_j \in \Omega^\circ$ and $\mathbf{g} = [g(\mathbf{x}_j)], \mathbf{x}_j \in \partial\Omega$. We note that a change in boundary conditions (11) is as simple as changing rows in matrix A in (13) as well as on the right–hand side $\mathbf{c}$ in (16).

3. Time discrete scheme

For discretization of the time variable, we need some preliminary. Let us consider

$$t_k = k\tau, \qquad k = 0,1, \dots, N_t,$$

where $\tau = \frac{T}{N_t}$ is the step size of the time variable.

3.1 The backward Euler method

In this section, we discrete the time variable using the backward Euler relation for the first–order derivative on the time variable. Consider Eq. (1) at time t_n, then we have

$$\frac{u^{n+1} - u^n}{\tau} = \Delta u^{n+1}, \qquad (17)$$

where

$$\Delta u^j = \frac{\partial^2 u^j(x,y)}{\partial x^2} + \frac{\partial^2 u^j(x,y)}{\partial y^2}, \qquad u^j(x,y) = u(x,y,t_j).$$

Simplifying Eq. (17) gives the following second–order PDE

$$u^{n+1} - \tau\Delta u^{n+1} = u^n, \qquad (18)$$

with the conditions (3)–(7) and the initial condition $u^0 = f(x,y)$.

3.2 Crank–Nicolson technique

In this section, we use the Crank–Nicolson technique [20]. Consider Eq. (1) at time $t_{n+1/2}$, then the central finite difference formula gives

$$\frac{u^{n+1}-u^n}{\tau} \simeq u_t^{n+\frac{1}{2}} = \Delta u^{n+\frac{1}{2}} \simeq \frac{1}{2}(\Delta u^{n+1} + \Delta u^n). \tag{19}$$

Simplifying Eq. (19) gives the following PDE

$$u^{n+1} - \frac{\tau}{2}\Delta u^{n+1} = \frac{\tau}{2}\Delta u^n + u^n. \tag{20}$$

with the conditions (3)–(7) and $u^0 = f(x,y)$.

4. Implementation of the procedure

The non–local boundary value problem is implemented by RBFs in this section. In order to obtain the unknown function $u(x,y,t)$, we approximate two–dimensional $u^n(x,y)$ by RBFs as

$$u^n \simeq \sum_{k=1}^{N} \lambda_k^n \phi(r_k), \tag{21}$$

$$\Delta u^n \simeq \sum_{k=1}^{N} \lambda_k^n \Delta\phi(r_k), \tag{22}$$

where $r_k = \sqrt{(x-x_i)^2 + (y-y_j)^2}, x_i = i/m_x, y_j = j/m_y$ and

$$\Delta\phi(r) = \frac{1}{r}\phi'(r) + \phi''(r).$$

Also according to the domain of the problem, we choose the uniform grid as collocation points [16]. Now applying Kansa's method gives the system of equations as follows

$$\begin{bmatrix} L & 0 \\ B & \tilde{h}_0^{n+1} \end{bmatrix} \begin{bmatrix} \Lambda^{n+1} \\ \mu^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \tilde{\mathbf{g}} \end{bmatrix}, \tag{23}$$

where the dimension of the coefficient matrix is $N+1 \times N+1$. Also, $B, \tilde{h}_0^{n+1}$ and $\tilde{g}$ are the same for Euler and Crank–Nicolson techniques and defined by

$$B = \begin{bmatrix} \dfrac{\partial\phi(0,y_j)}{\partial x} \\ \dfrac{\partial\phi(1,y_j)}{\partial x} \\ \phi(x_i,1) \\ \int_0^1\int_0^1 \phi(x,y)dxdy \\ \phi(x_i,0) \end{bmatrix}, \tilde{h}_0^{n+1} = \begin{bmatrix} 0 \\ -h_0^{n+1}(x_i) \end{bmatrix}, \tilde{g} = \begin{bmatrix} g_0^{n+1}(y_j) \\ g_1^{n+1}(y_j) \\ h_1^{n+1}(x_i) \\ m^{n+1} \\ 0 \end{bmatrix}, \quad j = 0,\dots,m_y, i = 1,\dots,m_x - 1.$$

And also in Euler method, we have

$$L = \Phi - \tau\Delta\Phi, \qquad \mathbf{f} = u^n.$$

and in Crank–Nicolson technique, one has

$$L = \Phi - \frac{\tau}{2}\Delta\Phi, \qquad \mathbf{f} = \frac{\tau}{2}\Delta u^n + u^n.$$

By solving the system of equations (23), the unknown λ_k and $\mu(t)$ will be obtained at any time step as well.

5. Numerical Results

In this section, in order to support our theoretical discussion, the proposed method is employed to obtain the solutions of two test problems. Accordingly, the following root mean square (RMS) error are applied to make a comparison

$$RMS = \frac{\sqrt{\sum_{k=1}^{M} (u(\tilde{\mathbf{x}}_k, T) - u_N(\tilde{\mathbf{x}}_k, T))^2}}{M},$$

where T is the final time and M is the number of test points $\tilde{\mathbf{x}}_k \in (0,1)^2$. In this paper, we have utilized 121 uniform nodes as test points.

Example 1 *Consider Eqs. (1)–(7) with*

$$f(x,y) = (1+y)\exp(-x), \qquad g_0(y,t) = -(1+y)\exp(t),$$
$$g_1(y,t) = -(1+y)\exp(t-1), \quad h_1(x,t) = 2\exp(t-x),$$
$$h_0(x) = \exp(-x), \qquad m(t) = \frac{3}{2}\exp(t)(1-\exp(-1)).$$

The exact solution of this problem is $u(x,y,t) = (1+y)\exp(t-x)$ and $\mu = \exp(t)$. Tables 1–1 show the RMS errors of the presented method at time $T = 0.2$ via four RBFs listed in Table 1.

Table 2: The RMS error for Example 1 using Backward Euler method with $c = 0.5$ and $T = 0.2$

N	GA		MQ	
	u	μ	u	μ
25	7.143514e − 4	1.547301e − 3	8.245392e − 4	2.008794e − 3
36	5.800419e − 4	2.592134e − 4	6.070198e − 4	3.891045e − 4
49	5.722920e − 4	2.488811e − 4	5.806599e − 4	3.430788e − 4
64	5.707776e − 4	2.238243e − 4	5.719409e − 4	2.287703e − 4
81	5.706766e − 4	2.236760e − 4	5.708401e − 4	2.251799e − 4

Table 3: The RMS error for Example 1 using Backward Euler method with $c = 0.5$ and $T = 0.2$

N	IMQ		IQ	
	u	μ	u	μ
25	1.499449e − 3	5.269310e − 3	2.078105e − 3	7.854362e − 3
36	7.344843e − 4	8.035454e − 4	8.688428e − 4	1.173190e − 3
49	6.094943e − 4	6.407271e − 4	6.399969e − 4	9.039736e − 4
64	5.777300e − 4	2.670414e − 4	5.835578e − 4	3.034872e − 4
81	5.729517e − 4	2.532407e − 4	5.749529e − 4	2.795253e − 4

Table 4: The RMS error for Example 1 using Crank–Nicolson method with $c = 0.5$ and $T = 0.2$

N	GA		MQ	
	u	μ	u	μ
25	7.136526e − 4	1.545805e − 3	8.237256e − 4	2.007224e − 3
36	5.797977e − 4	2.576134e − 4	6.067658e − 4	3.875977e − 4
49	5.720514e − 4	2.472979e − 4	5.804129e − 4	3.415761e − 4
64	5.705479e − 4	2.222662e − 4	5.717575e − 4	2.273245e − 4
81	5.704466e − 4	2.221170e − 4	5.706519e − 4	2.237368e − 4

Table 5: The RMS error for Example 1 using Crank–Nicolson method with $c = 0.5$ and $T = 0.2$

N	IMQ		IQ	
	u	μ	u	μ
25	1.497938e − 3	5.267270e − 3	2.076169e − 3	7.851984e − 3
36	6.391531e − 4	8.019499e − 4	8.677832e − 4	1.171528e − 3
49	6.091174e − 4	6.391531e − 4	6.395138e − 4	9.023498e − 4
64	5.775216e − 4	2.655530e − 4	5.833293e − 4	3.019794e − 4
81	5.727393e − 4	2.517548e − 4	5.747248e − 4	2.780210e − 4

Example 2 *Consider Eqs. (1)–(7) with*

$$f(x,y)=\cos(\pi x)\cos(\pi y),\quad g_0(y,t)=0,$$
$$g_1(y,t)=0,\qquad h_1(x,t)=-\exp(-2\pi^2 t)\cos(\pi x),$$
$$h_0(x)=\cos(\pi x),\qquad m(t)=0.$$

The exact solution of this problem is $u(x,y,t)=\exp(-2\pi^2 t)\cos(\pi x)\cos(\pi y)$ and $\mu=\exp(-2\pi^2 t)$. Tables 2–2 show the RMS errors of the presented method at time $T=1$.

Table 6: The RMS error for Example 2 using Backward Euler method with $T=1$

N	GA with $c=2.5$		MQ with $c=1.5$	
	u	μ	u	μ
25	2.435434e − 4	9.047488e − 4	1.237610e − 4	5.624211e − 3
36	1.834072e − 4	9.016421e − 4	8.972573e − 4	3.509063e − 3
49	1.010841e − 4	1.216578e − 4	2.168052e − 4	7.975513e − 4
64	5.092937e − 5	1.044367e − 4	1.569789e − 4	6.744631e − 4
81	8.361560e − 6	2.310227e − 5	2.717235e − 5	1.282102e − 4

Table 7: The RMS error for Example 2 using Backward Euler method with $c=2.5$ and $T=1$

N	IMQ		IQ	
	u	μ	u	μ
25	8.725659e − 4	3.499866e − 3	6.401000e − 4	1.307878e − 3
36	6.142274e − 4	2.906402e − 3	1.749520e − 4	3.891655e − 4
49	6.070773e − 4	1.737066e − 3	1.322309e − 4	3.336629e − 4
64	1.440856e − 4	8.639197e − 4	3.213281e − 5	1.061316e − 4
81	1.401112e − 4	3.762950e − 4	2.337347e − 5	2.341166e − 5

Table 8: The RMS error for Example 2 using Crank–Nicolson method with $T=1$

N	GA with $c=2.5$		MQ with $c=1.5$	
	u	μ	u	μ
25	3.015771e − 4	7.634608e − 4	5.500958e − 4	1.673105e − 3
36	2.820398e − 4	4.872980e − 4	1.708277e − 4	6.385230e − 4
49	1.997431e − 4	3.144910e − 4	6.889863e − 5	3.985817e − 4
64	5.952093e − 5	2.252592e − 4	5.341752e − 5	3.393935e − 4
81	2.805438e − 5	1.628911e − 4	3.285364e − 5	8.657550e − 5

Table 9: The RMS error for Example 2 using Crank–Nicolson method with $c=2.5$ and $T=1$

N	IMQ		IQ	
	u	μ	u	μ
25	6.721957e − 4	2.574244e − 3	4.524163e − 4	9.120709e − 4
36	4.106373e − 4	2.294625e − 3	3.670923e − 4	8.452119e − 4
49	2.639083e − 4	1.684606e − 3	2.862322e − 4	5.386827e − 4
64	4.305403e − 5	2.140781e − 4	1.714935e − 4	4.039737e − 4
81	3.330737e − 5	1.282475e − 4	1.128615e − 4	3.839403e − 4

6. Conclusion

This paper develops a numerical method for solving the time–depended diffusion equation with non–local and mixed Neumann–Dirichlet boundary conditions. For this sake, two well–known finite difference schemes, the backward Euler and Crank–Nicolson, has been applied for the time–discretization. The spatial derivatives and non–local and mixed boundary conditions have been thoroughly satisfied via RBFs. The numerical results demonstrate the efficiency and reliability of the proposed method.

References

[1] A. Alipanah and S. Esmaeili. "*Numerical solution of the two–dimensional Fredholm integral equations using Gaussian radial basis function"*. J. Compute. Appl. Math. Vol. 235, pp. 5342-5347. (2011).

[2] A. E. Tarwater. "*A parameter study of Hardy's multiquadric method for scattered data interpolation"*. Report UCRL-53670, Lawrence Livermore National Laboratory. (1985).

[3] D. Mirzaei and M. Dehghan. "*A meshless based method for solution of integral equations*". Appl Numer Math. Vol. 60, pp. 245-262. (2010).

[4] EJ. Kansa. "*Multiquadrics-a scattered data approximation scheme with applications to computational fluid–dynamics-I surface approximations and partial derivative estimates"*. Math. Comput. Appl. Vol. 19, pp. 127-145. (1990).

[5] EJ. Kansa. "*Multiquadrics-a scattered data approximation scheme with applications to computational fluid–dynamics-II. Solutions to parabolic, hyperbolic and elliptic partial differential equations*". Math. Comput. Appl. Vol. 31, pp. 147-161. (1991).

[6] G. Fairweather, R.D. Saylor. "*The reformulation and numerical solution of certain nonclassical initial–boundary value problems*". SIAM J. Sci. Statist. Comput. Vol. 12, pp. 127-144. (1991).

[7] J.R. Cannon, Y. Lin, A.L. Matheson. "*The solution of the diff€usion equation in two–space variables subject to the specification of mass*". J. Appl. Anal. Vol. 50, pp. 1-19. (1993).

[8] M. Dehghan and M. Tatari. "*Determination of a control parameter in a one–dimensional parabolic equation using the method of radial basis functions"*. Math. Comput. Modell. Vol. 44, pp. 1160-1168. (2006).

[9] M. Dehghan and M. Tatari. "*Use of radial basis functions for solving the second order parabolic equation with non–local boudary conditions"*. Numer. Meth. Partial Diff. Equ. Vol. 24, pp. 924-938. (2008).

[10] M. Dehghan. "*Second–order schemes for a boundary value problem with Neumann boundary conditions*, J. Comput. Appl. Math. Vol. 138, pp. 173-184. (2002).

[11] Q. Liu, Y. Gu, P. Zhuang, F. Liu and Y. Nie. "*An implicit RBF meshless approach for time fractional diffusion equations"*. Comput Mech. Vol. 48, pp. 1-12. (2011).

[12] R. Franke. "*Scattered data interpolation: test of some methods*". Math. Comput. Modell. Vol. 38, pp. 181-200. (1982).

[13] R. Salehi and M. Dehghan. "*A moving least square reproducing polynomial meshless method"*. Appl Numer. Math. Vol. 69, pp. 34-58. (2013).

[14] S. Abbasbandy, A. Shirzadi. "*MLPG method for two–dimensional diffusion equation with Neumannâ€™s and non–classical boundary conditions"*. Appl. Numer. Math. Vol. 61, pp. 170-180. (2011).

[15] S. Abbasbandy, A. Shirzadi. "*A meshless method for two-dimensional diffusion equation with an integral condition"*. Engineering Analysis with Boundary Elements, Vol. 34, No. 12, pp. 1003-1120. (2010).

[16] S. Kazem and J. A. Rad. "*Radial basis functions method for solving of a non–local boundary value problem with Neuman's boudary conditions"*. Appl. Math. Modell. Vol. 36, pp. 2360-2369. (2012).

[17] S. Wang, Y. Lin. "*A finite–difference solution to an inverse problem determining a control function in a parabolic partial differential equations"*. Inverse Problems. Vol. 5, pp. 631-640. (1989).

[18] V. Capsso, K. Kunisch. "*A reaction–diffusion system arising in modeling man–environment diseases*". Q. Appl. Math. Vol. 46, pp. 431-449. (1988).

[19] YT. Gu, P. Zhuang and F. Liu. "*An advanced implicit meshless approach for the non–linear anomalous subdiffusion equation*". Comput Model Eng Sci. Vol. 56, pp. 303-334. (2010).

[20] G. Evans, J. Blackledge, P. Yardley. "*Numerical methods for partial differential equations"*. Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London, (2000).

[21] Mesgarani Hamid, Azari Yaqub. *"Numerical investigation of Fredholm integral equation of the first kind with noisy data"*. Mathematical Sciences. Vol. 13, No. 3, pp. 267-278. (2019).

[22] Aghdam Yones Esmaeelzade, Safdari Hamid, Azari Yaqub, Jafari Hossein. *"Baleanu Dumitru Numerical investigation of space fractional order diffusion equation by the Chebyshev collocation method of the fourth kind and compact finite difference scheme"*. Discrete and Continuous Dynamical Systems-S. (2018).

[23] Azari Yaqub, Garmanjani Gholamreza, Rabiei Hamed. *"Numerical of nonlinear Sine-Gordon equation with local RBF-based finite difference collocation method*". (2013).

[24] Rashidinia Jalil, Azari Yaqub, Garmanjani Gholamreza. *"Numerical solution of the nonlinear Fredholm integral equations of the second kind by radial basis functions"*. (2012).

[25] K. Shanazari and S. Banei. *"A meshfree method with a non-overlapping do- main decomposition method based on TPS for solving the forward-backward heat equation in two-dimension"*. Numerical Algorithms. Vol. 85, No. 1, pp. 1-21. (2020).

[26] S. Banei, and K. Shanazari. *"Solving the forward-backward heat equation with a non-overlapping domain decomposition method based on multiquadric RBF meshfree method"*. Computational Methods for Differential Equations. (2020).

[27] S. Banei, and K. Shanazari. *"On the convergence analysis and stability of the RBF-adaptive method for the forward-backward heat problem in 2D"*. Applied Numerical Mathematics. Vol. 159, No. 0, pp. 297-310. (2020).

[28] S. Banei, and K. Shanazari. *"A non-iterative domain decomposition method for the forward-backward heat equation*". Iranian Journal of Numerical Analysis and Optimization. Vol. 10, No. 2, pp. 155-167. (2020).